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AUTONOMOUS CAR MOTION PLANNING USING PATH APPROXIMATION AND GEOMETRIC CONTROL ALGORITHMS

Summary. The paper presents control algorithms in the path following task. An important step in planning the motion of an autonomous vehicle is the initial definition and description of the route. In this study, the route has been approximated using the spline function B3. The paper presents a comparison of the effectiveness of the control algorithms that are applied to determine steering angle. Our own algorithm B3M is formulated, and its effectiveness is compared with classical ones. The proposed algorithm has been developed on the basis of a model with 3 degrees of freedom (3DoF) and it can be used in combination with more complex vehicle dynamic models, such as those with 5, 7 and 10 degrees of freedom. After implementing computer models of vehicle dynamics with 3, 5, 7 and 10 DoF, they were verified and validated. The computer simulation results presented in this paper confirmed the correctness of the models and the proposed B3M steering algorithm. This algorithm does not require the declaration of constants and is as effective as other geometrical algorithms such as Pure Pursuit.

Keywords: autonomous vehicle, motion control, vehicle dynamic models, path following, steering algorithm

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One important step in planning the movement of an autonomous vehicle is the description of its trajectory (displacement of the vehicle's center of mass). This process, which ends in trajectory realization, starts with path planning (Fig.1).

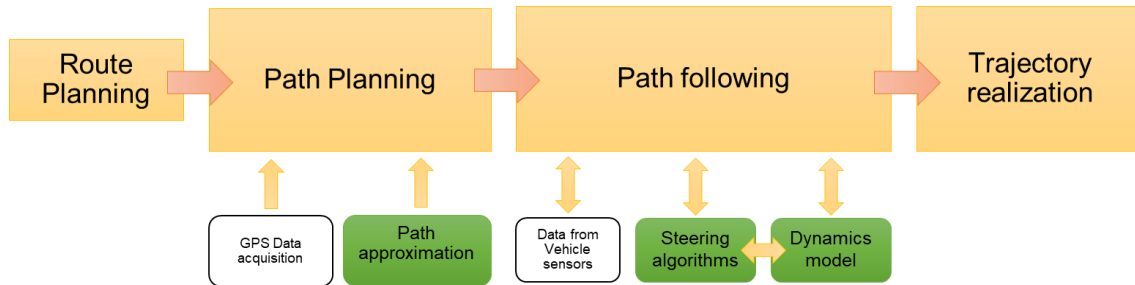


Fig. 1. Motion planning of an autonomous vehicle

Appropriate planning of the movement path has a decisive impact on the subsequent path following task. There are many methods of determining the path [1]. Among the frequently used methods are polynomial approximation, spline functions, Bézier curves, and creating a path by connecting points using arcs or clothoids. All methods aim to generate the smoothest possible path [2, 3, 4, 5]. Geometric control algorithms, commonly used in autonomous vehicle path following tasks, are dependent on the path curvature; therefore, the accuracy of path determination is very important. The implementation of the path using control algorithms is presented in the paper [6]. It consists of finding such a steering angle of the front wheels that, with a known speed course, the center of mass of the vehicle is as close to the path as possible. The literature distinguishes such control methods as: Linear Quadratic Regulator [7], Linear Model Predictive Control [8], or classic geometric algorithms such as Pure Pursuit [9]. Dynamic optimization for determining the steering angle has been considered in [10, 11]. A review of the optimization methods and path approximation can also be found in [12, 13].

2. DYNAMIC MODELS OF A VEHICLE

Choosing an appropriate dynamics model is a difficult task. One problem that has to be solved is that numerically efficient vehicle dynamics models tend to be highly simplified and thus inadequately represent vehicle behavior in reality. On the other hand, complicated vehicle models better describe the vehicle motion but tend to be very demanding in computation time, which does not make them feasibly applicable to autonomous car control [9]. The literature is dominated by models with a few (3-14) DoF, and therefore containing many simplifying assumptions. The most popular is the model with 3 DoF, also known as the Bicycle Model (BM) [14, 15, 11]. A review of the papers related to trajectory control (covering works up to 2016) is given in the paper [16]. In paper [14], the authors also present a control algorithm based on a vehicle model with 3 DoF. In addition, they take into account the possibility of obstacle avoidance and the variable velocity of the vehicle. The papers [1] and [17] review works and methods related to motion planning and control again using models with 3 DoF. More complex vehicle models are less common than the bicycle model and its variations. In [18], a planar model with 5 DoF is presented. In addition to the three degrees of freedom that describe the body's motion, the angles of rotation of the front and rear wheels are also included. This enables the introduction of the driving and braking moments into the considerations, as well as the application of the tire model to describe the road-wheel interaction forces. Models

with 3 DoF and 9 DoF are presented in [15], and models with 10 and 14 degrees of freedom in [17]. A model with 10 degrees of freedom is presented in paper [11], which uses a homogeneous transformation method to formulate the vehicle dynamics model. In modeling the dynamics of an autonomous car, the calculation time is an important factor.

The vehicle dynamics models in Fig. 2 can be formulated with varying complexity. They can be either planar or spatial models with varying degrees of freedom.

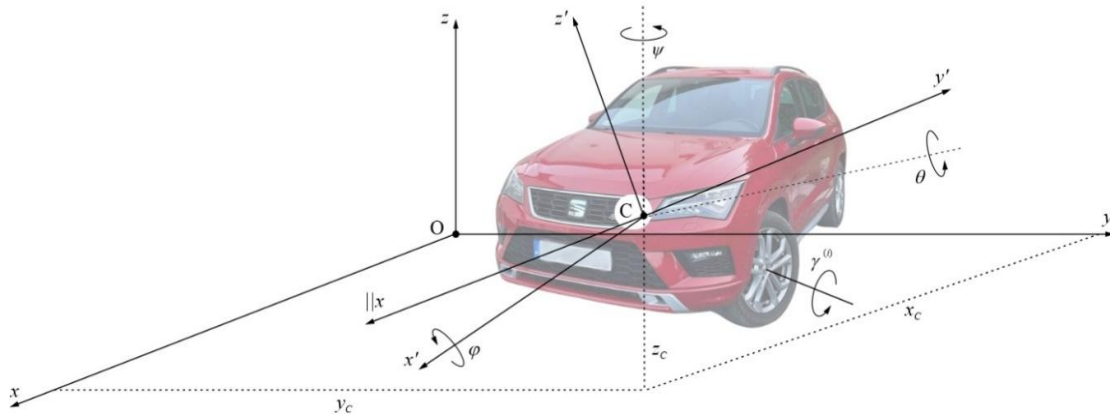


Fig. 2. General scheme of the vehicle

In the classical planar model with 3 degrees of freedom, the model is formulated in a local coordinate system (connected with the body's center of gravity {C}). The model formulated in this coordinate system requires additional calculations when determining the displacement of the center of gravity in the inertial coordinate system { }. In this paper, the bicycle model is formulated in the inertial system { } associated with the roadway. The short description of the vehicle dynamics models considered and implemented are presented in Tab. 1.

Tab. 1

Characteristics of the vehicle dynamics models considered

| 10 DoF | 7 DoF | 5 DoF | 3 DoF |
|--|---|---|---|
| $\mathbf{r}_c = \begin{bmatrix} x_c \\ y_c \\ z_c \end{bmatrix}$ displacements of the body's center of mass $\mathbf{B} = \begin{bmatrix} \psi \\ \theta \\ \varphi \end{bmatrix}$ body rotation angles $\mathbf{K} = \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \\ \gamma^4 \end{bmatrix}$ wheel rotation angles | $\mathbf{r}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$ displacements of the body's center of mass $\mathbf{B} = \psi$ yaw angle $\mathbf{K} = \begin{bmatrix} \gamma^1 \\ \gamma^2 \\ \gamma^3 \\ \gamma^4 \end{bmatrix}$ wheel rotation angles | $\mathbf{r}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$ displacements of the body's center of mass $\mathbf{B} = \psi$ yaw angle $\mathbf{K} = \begin{bmatrix} \gamma^{1,2} \\ \gamma^{3,4} \end{bmatrix}$ angles of rotation of the substitute wheels | $\mathbf{r}_c = \begin{bmatrix} x_c \\ y_c \end{bmatrix}$ displacements of the body's center of mass $\mathbf{B} = \psi$ yaw angle |

The 10-degree-of-freedom model is the most accurate of those presented in this paper. The equations of motion for this model were formulated based on the Newton-Euler equations. To determine the roadway response forces (reactions) to the vehicle wheels, the modified Pacejka & Sharp Brush Tire Model [9] was used.

The 7-degree-of-freedom model, although formally a planar, allows angles of roll φ and pitch θ to be taken into account, as well as vertical displacement z_c , when determining the roadway normal reactions on the vehicle wheels. These additional coordinates are determined in a quasi-static manner and are not generalized coordinates.

The 5-degree-of-freedom model is an extension of the 3 DoF model, including 2 degrees of freedom to determine the angles of rotation of the fictitious front and rear wheels. The reactions of the roadway interaction on the fictitious vehicle wheels are determined as in the 7 DoF and 10 DoF models.

In order to verify the models proposed, results were compared using the CarSim software. Vehicle (A-CLASS HATCHBACK from CarSim dataset) velocity was 60 km/h, amplitude of the steering angle varied from $\delta = 1.32^\circ$, to 0.4° at the end (Fig. 3a). All simulations were performed on a flat road, taking the coefficient of adhesion $\mu = 0.8$. The fourth-order Runge-Kutta method with a constant integration step was used to integrate the non-linear equations of vehicle motion. The trajectories obtained can be seen in Fig. 3b.

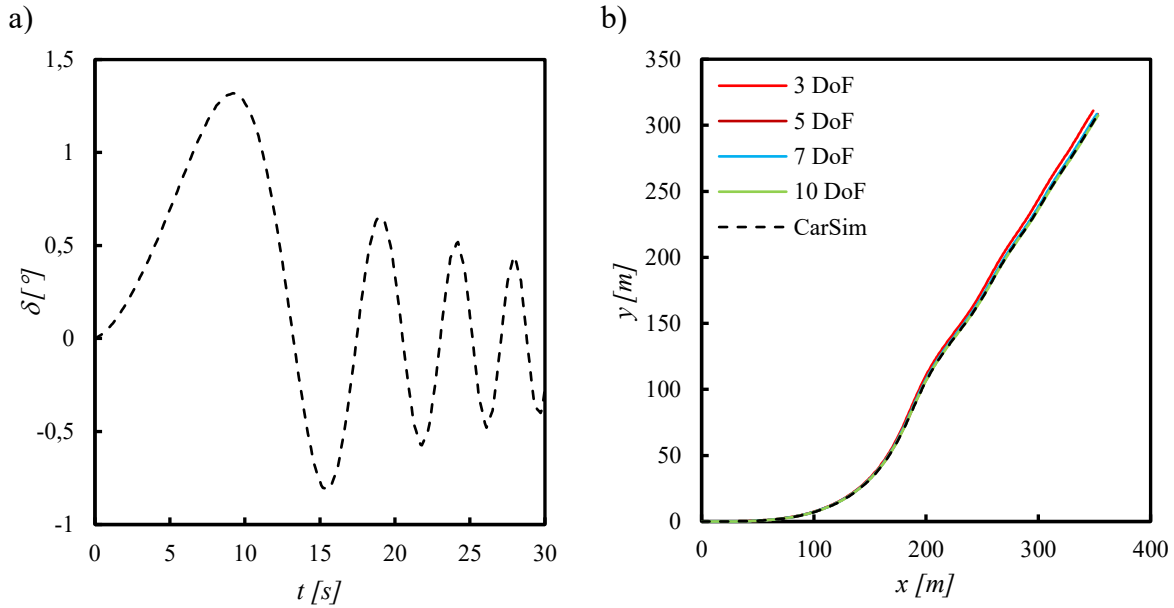


Fig. 3. Comparison with CarSim
a) Wheel steering angle course; b) Vehicle trajectory

All the models give results consistent with CarSim. Error (displacement: x , y and yaw angle ψ) did not exceed 1%. Tab. 2. presents the numerical effectiveness of the models.

The most compliant with CarSim was the 10 degrees of freedom model, but it can be seen that the model with 3 DoF is by far the most numerically efficient. It is almost 35 times faster than the model with the highest complexity (10 DoF).

The proposed models were also validated by comparing the calculation results obtained using our own models with the results of a real measurement experiment recorded on the track

(CERAM) located in Mortefontaine, France. This experiment is presented in [19]. A graphical representation of the simulation results (trajectory) is presented in Fig. 4.

Tab. 2

Numerical effectiveness of considered models

| Model (DoF) | Integration step | Calculation time [s] |
|-------------|------------------|----------------------|
| 3 | 0.010 | 0.008 |
| 5 | 0.004 | 0.033 |
| 7 | 0.004 | 0.111 |
| 10 | 0.004 | 0.276 |

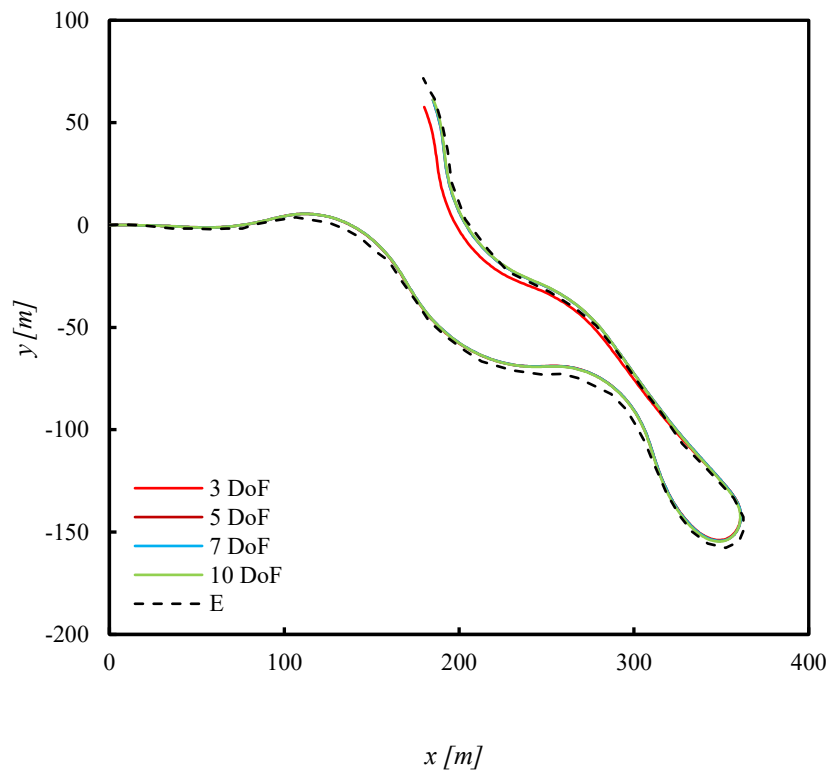


Fig. 4. Vehicle trajectory comparison with experiment (E)

Due to limited data, only the yaw angle ψ could be compared. The error did not exceed 2%. Verification and validation of the presented models indicate that they are compatible and can be used in the path following task.

3. PATH PLANNING

The function describing steering angle is directly related to the calculation of path radius of curvature necessary in all the steering angle control algorithms considered in this paper, which are based on geometric relationships. The set of points $(x_0^e, y_0^e) \dots (x_m^e, y_m^e)$ describing the route should be sufficiently dense, especially for path with high curvature. If the route is given in a discrete way, it is necessary to have an approximation function that allows a continuous representation of the path. Low approximation accuracy would result in large errors in

the representation of the path. In this paper, approximation was realized using the B_3 spline functions. The functions describing the desired displacements of the vehicle's center of gravity are assumed to be of the form:

$$x(s) = \sum_{i=-1}^{n+1} a_i^x \varphi_i(s) \quad (2)$$

$$y(s) = \sum_{i=-1}^{n+1} a_i^y \varphi_i(s) \quad (3)$$

where: a_i – coefficients, n – number of subdivisions into which the road segment $< A, B > \equiv < x_0^e, x_m^e >$ is divided, s – the distance travelled by the vehicle.

Fig. 5 shows the base function $\varphi_i(s)$.

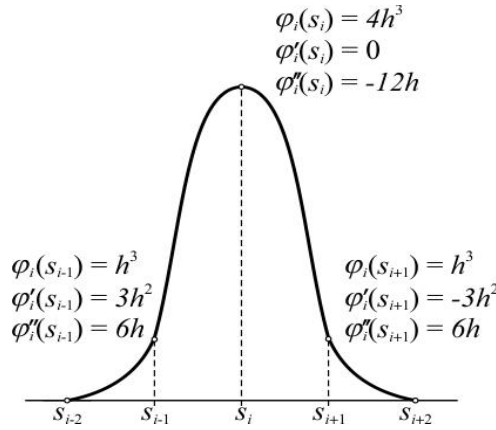


Fig. 5. Function $\varphi_i(s)$

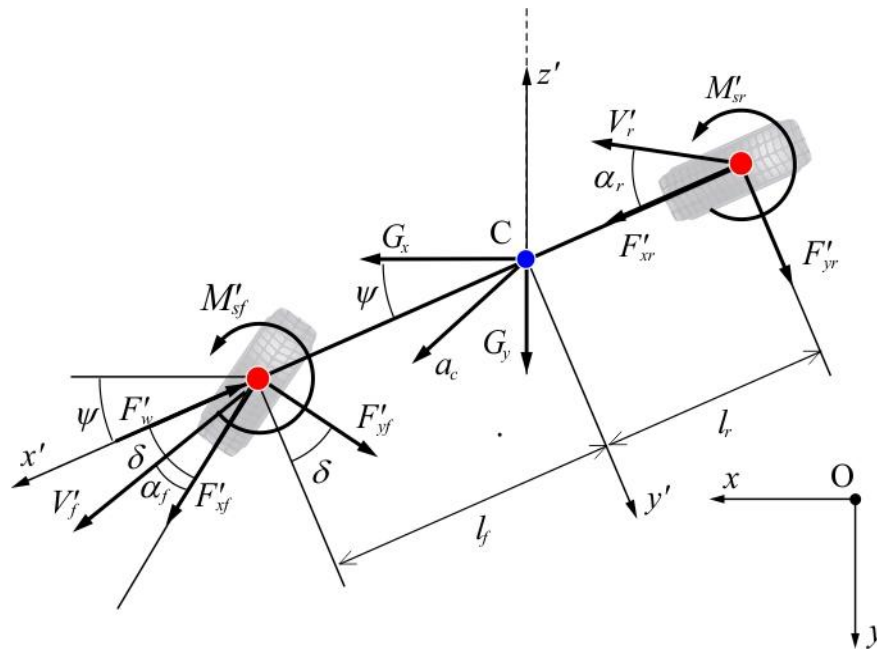
The coefficients a_i^x, a_i^y , present in (2) and (3) (for $i = -1, 0, 1 \dots n, n+1$), are selected from the condition of minimizing functionals:

$$\Omega_x(a_{-1}^x \dots a_{n+1}^x) = \sum_{i=0}^m [x(s_i^e) - x_i^e]^2 \rightarrow \min \quad (4)$$

$$\Omega_y(a_{-1}^y \dots a_{n+1}^y) = \sum_{i=0}^m [y(s_i^e) - y_i^e]^2 \rightarrow \min \quad (5)$$

$$s_k^e = \sum_{j=1}^k \sqrt{[(x_j^e - x_{j-1}^e)^2 + (y_j^e - y_{j-1}^e)^2]}. \quad (6)$$

The author's algorithm proposed in this paper is based on the equations of motion of a model with 3 DoF formulated in an inertial coordinate system. The algorithm is hereafter referred to as B3M. A schema of the vehicle model is shown in Fig. 6.



The generalized coordinates are taken as x_c, y_c, ψ , and the equations of motion are:

$$I_z' \ddot{\psi} = -F_{yr}' l_r + F_{xf}' l_f s \delta + F_{yf}' l_f c \delta + M_{sr}' + M_{sf}', \quad (9)$$

where:

m – total mass of the vehicle (including wheels),

I'_z – vehicle mass moment of inertia about axle z' ,

x_c, y_c – coordinates of the center of gravity of the vehicle,

ψ – yaw angle,

$F'_{xf}, F'_{yf}, F'_{xr}, F'_{yr}$ – the components of the road forces,

δ – steering angle of the front wheels,

l_r, l_f – the distance of the rear and front axles of the vehicle from its center of gravity,

F'_w – drag force,

M'_{sr}, M'_{sf} – self-aligning moments defined in [9],

G_x, G_y – components of the gravitational force.

Lateral forces F'_{yf}, F'_{yr} can be defined as in [9]:

$$F'_{yf} = -C_{\alpha_f} \alpha_f, \quad (10)$$

$$F'_{yr} = -C_{\alpha_r} \alpha_r, \quad (11)$$

where: $C_{\alpha_f}, C_{\alpha_r}$ – the lateral stiffness coefficients of both wheels of the vehicle,

α_f, α_r – slip angles.

Slip angles α_r i α_f can be determined from the relation:

$$\alpha_r = \frac{V'_{yr}}{V'_{xr}}, \quad (12)$$

where: $V'_{x,r} = V'_{xc}$; $V'_{yr} = V'_{yc} - \dot{\psi} l_r$,

and:

$$\alpha_f = -\delta + \frac{V'_{y,f}}{V'_{x,f}}, \quad (13)$$

where: $V'_{xf} = V'_{xc}$; $V'_{yf} = V'_{yc} + \dot{\psi} l_f$, V'_{xc}, V'_{yc} are components of the velocity of the center of gravity in the coordinate system $\{C\}'$.

In the cases analyzed, it was assumed that the tangential velocity of the vehicle V'_x is a known function of time:

$$V'_x = V'_x(t) = V'_{nom}, \quad (14)$$

The driving force was defined using a proportional controller law:

$$F'_N = F'_{xf} = k(V'_{xc} - V'_{nom}). \quad (15)$$

From equations (11) can be obtained:

$$\delta = \frac{1}{C_{\alpha_f}} [F'_{yf} + F'_{yr} + C_{\alpha_f} \theta_f + C_{\alpha_r} \theta_r] \quad (16)$$

The algorithm proposes a special iterative procedure for determining the $\delta(t)$. It is worth noting that the data for determining $\delta(t)$: $x_T, y_T, \dot{x}_T, \dot{y}_T, \ddot{x}_T, \ddot{y}_T$ are taken from the path approximating function, and also the velocity course $V'_x(t)$ is known. On the other hand $\psi(t^-), \dot{\psi}(t^-), \ddot{\psi}(t^-)$ can be provided from the vehicle dynamics model (any). The whole process can be seen in Fig. 7.

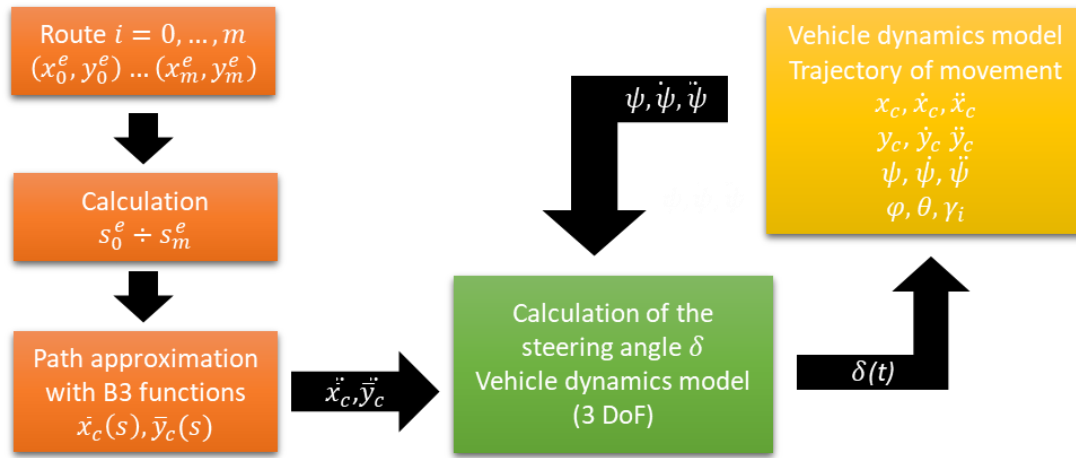


Fig.7. Schema of proposed algorithm B3M

4. RESULTS AND DISCUSSION

The implemented route had a large curvature and the vehicle travelled at a variable velocity (J-Turn). The velocity course is shown in Fig. 8a, and the trajectories in Fig. 8b. The vehicle velocity for this maneuver was reconstructed on the basis of the actual location of road signs along the route, which is an entrance to the Bielsko-Biała (Poland) highway.

The average approximation error for the analyzed route was 0.06 m. The calculations assumed number of points $m = 180$ and number of approximation functions $\varphi_i((s)t), n = 60$. The parameter (k, l_d) needed for the Stanley and Pure Pursuit algorithms was set to 6 and 0.05. Fig. 9a,b shows the steering angle course calculated by the different algorithms.

The course of the functions representing the steering angle obtained using all the algorithms is similar (Fig. 9a). The function curves representing the steering angle obtained with the B3M, Pure Pursuit, and Stanley Control algorithms are very similar (Tab. 3). In this particular case, where uncomplicated geometric algorithms were used for control, the values obtained are within the generally accepted norm (except for the Rajamani algorithm). The calculation time from the reading of the points by approximation to the calculation of the specific movement trajectory does not exceed 1 second for any of the compared algorithms.

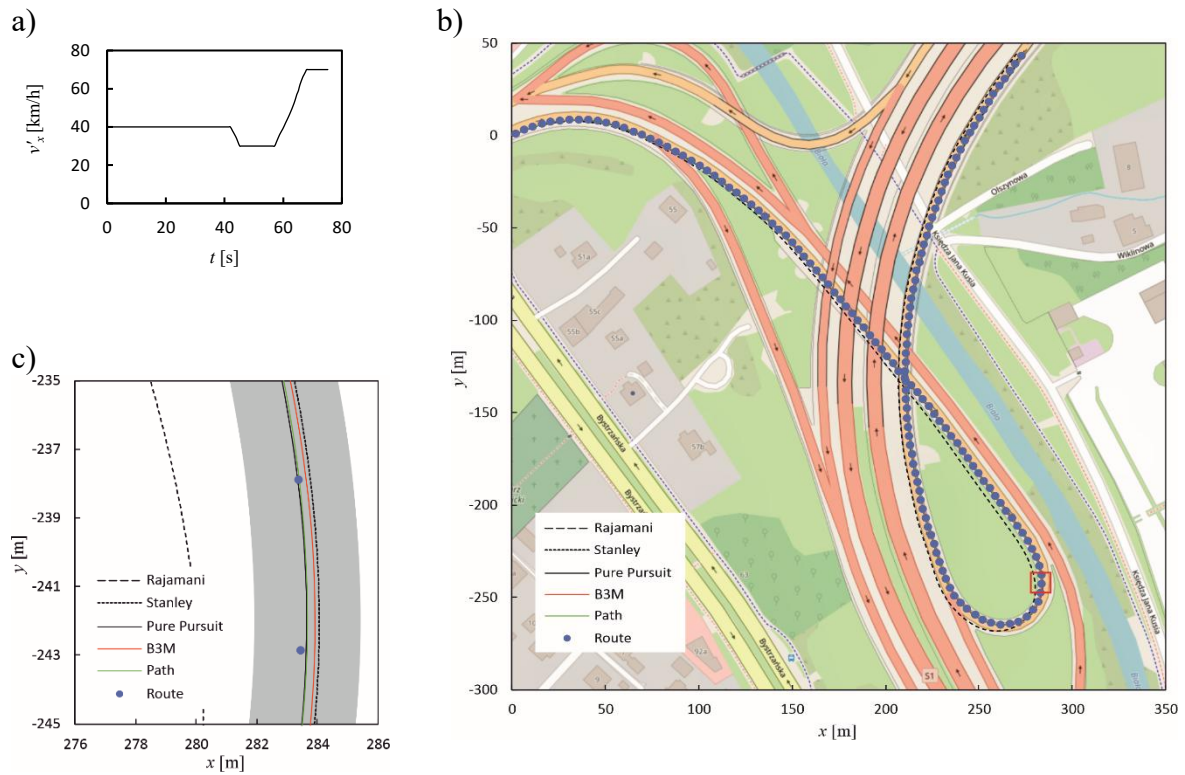


Fig. 8. J-Turn route a) vehicle velocity course V'_x ; b) Desired path and trajectory determined using different algorithms c) enlargement

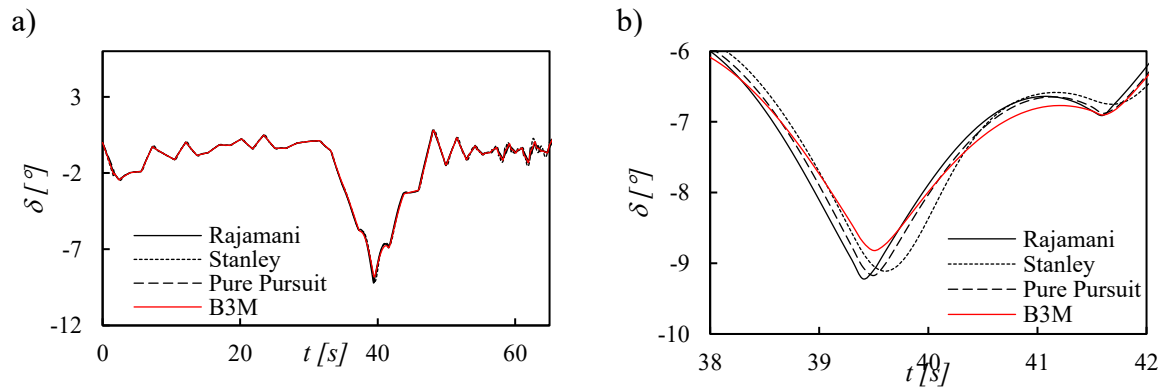


Fig. 9. Steering angle course: a) $t \in [0; 67s]$, b) $t \in [38; 42s]$

Tab. 3

Path error for different algorithms

| Algorithm | Path error Δ_{max} [m] | Constant k, l_d |
|-----------------|-------------------------------|-------------------|
| B3M | 0.629 | — |
| Pure Pursuit | 0.369 | 0.05 |
| Stanley Control | 0.382 | 6 |
| Rajamani | 6.404 | — |

4. CONCLUSIONS

The dynamics models presented in this paper make it possible to model vehicle motion dynamics with an accuracy close to that offered by the commercial software. It is important to underline that the models presented can be extended or modified and their computer implementation can be easily changed. ,

The results of the validation and verification presented in this paper indicate that all the models presented are correct. The model with 10 degrees of freedom is the most accurate, and the model with 3 is the least accurate. At the same time, it is the most simplified model that is the most numerically efficient

The biggest disadvantage of more advanced models is their insufficient numerical efficiency. It should be noted that with the increase in the computational capabilities of processors, this disadvantage will lose its significance.

Approximation of the considered route to a path using B_3 spline functions generates an average error of 0.06 m. Taking into account the length and curvature of the approximated route, this result can be accepted as good.

The computational results presented in this paper made it possible to compare geometric steering algorithms and their application with vehicle dynamics models of varying degrees of freedom (complexity). The validation performed indicates that more complex vehicle dynamics models better represent actual traffic conditions. The proposed B3M algorithm showed very high accuracy. A comparison of the results for the different algorithms indicates that the proposed algorithm is correct and may be used in the path following task. The main advantages of the B3M algorithm include:

- no need to specify additional parameters, which are necessary when using the Stanley and Pure Pursuit algorithms,
- low dependence on the number of subintervals n (for path approximation functions),
- the possibility of combining it with models of different complexity (with 3, 5, 7 and 10 DoF),
- high numerical efficiency.

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