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STRESSES IN COMPOSITE PLATES WITH RIVETED BARS USED FOR AIRCRAFT CONSTRUCTION

Summary. Reinforcement of plates with rod systems is widely used in engineering, especially in aircraft construction. For example, An-178 aircraft. Removable panels on the lower surface of the wing half, located between the rear spar and the flaps. The method of calculating stresses and deformations in composite plates reinforced with rods is developed in the work. It is assumed that the rod is elastic, attached with rivets. Rivets were considered as rigid inclusions to which unknown forces were applied. These forces were determined from the condition of compatibility of plate and rod deformations. The singular integral equations' method was used to determine stresses and strains in the plate. Integral equations were solved numerically and reduced to a system of algebraic equations. To obtain the forces and moments acting on the rivets, the equations of equilibrium of the rivets and the conditions that ensure the same displacements between the rivets in the plate and in the rods are added to these equations. Examples of calculating stresses near circular and elliptical rivets, magnitudes of forces acting

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on rivets depending on the rigidity of the rods are given. The reduction of the stress concentration near the elliptical hole, which is placed between two rods, was studied.

Keywords: stress concentration factors (SCF), anisotropic plates, elastic rods, rivets, rigid inclusions, stress-strain state (SSS), stress concentration

1. INTRODUCTION

Reinforcement of plates by rod systems is widely used in engineering, especially in aircraft construction. For example, An-178 aircraft. Removable panels on the lower surface of the wing half, located between the rear spar and the flaps.

The study of stresses near a glued semi-finite rod (stiffening ribs) was investigated in [1]. In work [2], a wedge with a rod soldered to the boundary is considered. In work [3], the problem is considered in a more precise formulation (the rod was studied based on the elasticity). A soldered rod of finite length at the boundary of the half-space was considered in [4].

The paper examines the scenario, which is common in practice, especially in aircraft construction, when rods and plates are joined by rivets. A limited number of works are devoted to the study of the stress-strain state of such plates in the scientific literature. For the most part, such studies were performed for isotropic materials. In [5], a method for calculating stresses near a semi-infinite rod attached by a system of rivets is proposed. In [6], an approximate approach to the study of stresses and displacements in the plate is suggested, which is applicable in the case of large distances between rivets.

The system of periodically placed rivets was considered in [6]. In this paper, such problems are considered for composite, anisotropic materials.

An overview of works performed out until 2013 on the study of stresses near inclusions in isotropic and anisotropic plates is presented in [7]. The concerns for inclusion in isotropic plates were considered by analytical methods. In [8], the stresses near polygonal inclusions and in [9] - near inclusions of elliptical shape were investigated using this method. In [10], the inclusion of an arbitrary form is considered. The issue was solved using the theory of functions of a complex variable in the form of series.

The Eshelby problem, which considers problems with inclusions, was studied. In [11], the problem of an elliptical inclusion, which is inserted into a hole with tension, is considered. In [12], it is additionally assumed that a plate with an elliptical inclusion is under the action of a polynomial load at infinity. The Eshelby problem for inclusions of arbitrary shape was considered in [13] using a conformal mapping.

Much fewer works are devoted to the tasks of determining stresses in anisotropic plates with inclusions. In [14], a general method of studying anisotropic materials using the boundary element method is described. In [15], stress in a paraboloidal inclusion in an anisotropic plane was investigated by the analytical method. Experimental studies in polymer materials are given in [16].

The method of integral equations is widely used to study stresses near inclusions. To obtain integral equations for anisotropic plates, the Somiliano identity. The general approach to constructing such equations is obtained in [17]. Integral equations and numerical algorithms for solving them based on the boundary element method are described in the book [18]. The Isogeometric Boundary Element method is described in [20]. The Stroh formalism is used in Stroh formalism [20]. Other methods are also used, e.g., [21].

In this work, singular integral equations are used. These equations are obtained based on Lekhnitskiy's method and Cauchy's theorem.

In [22], the integral equations are obtained for anisotropic plates with rigid inclusions.

The effectiveness of this method is illustrated in [22] when solving a wide range of problems. Based on this method, a system of equations is obtained. One contains unknown forces and moments that are applied to inclusions (rivets). To find them, the conditions of compatibility of deformations of the plate with inclusions and rods are additionally considered.

2. FORMULATION OF THE PROBLEM

Consider an anisotropic plate reinforced by rods, which is under the action of tension at infinity. Let's accept; the rod is in tension-compression and bending conditions; there is no friction between the plate and the rods; a plane stress state occurs in the plate; rivets are considered rigid.

2.1. Basic relations for a plate with rigid inclusions

Let the elastic plate contain inclusions bounded by contours $L_1, \ldots, L_J$,, which do not intersect. The outer boundary of the plate is denoted by L_0

The plate should be set to the Cartesian coordinate system. The centers of gravity of the inclusions should be denoted C_j by $\left(c_j, d_j\right)$, $j=1,\ldots,J$. On the boundaries of inclusions L_j displacements are set (g_1, g_2), which have the form

$$g_1=\delta_j-\omega_j y,\ g_2=\gamma_j+\omega_j x,\ j=1,\ldots,J \qquad (1)$$

where $\delta_j, \gamma_j, \omega_j$ are constants, which characterize the displacement and rotation of each of the inclusions. As a result of the interaction of the plate and the rods, forces will be applied to the inclusions (rivets). Denote by (X_j, Y_j) and M_j the main vector and moment of these forces relative to the origin of coordinates, $j=1,\ldots,J$. Main moments M_j are considered given when considering problems with previously unknown angles of rotation of inclusions ω_j (at given turning angles, moments are determined in the process of solving the problem). Let us assume that the plate is under the influence of a load that acts at infinity.

2.2. Governing equations

Starting from Lekhnitskiy complex potentials $\Phi(z_1),\ \Psi(z_2)$, where $z_j=x+s_j y$, and $s_j,\ j=1,2$ are roots with positive imaginary part of characteristic equation [23]

$$\Delta(s)=\alpha_{11}s^4-2\alpha_{16}s^3+(2\alpha_{12}+\alpha_{66})s^2-2\alpha_{26}s+\alpha_{22}=0\,,$$

where α_{ij} are elastic compliances which are included in the Hooke's law.

Consider an arbitrary path Γ, which belongs to the domain D occupied by the plate. The tractions (X,Y) and displacements (u,v) are determined on this path by the formulas [22, 23].

On an arbitrary curve Γ, which belongs to the domain D stress vector (X,Y) and derivatives of displacements are determined by formulas (u,v)

$$Y=-2\operatorname{Re}\left[\Phi(z_1)z_1^{'}+\Psi(z_2)z_2^{'}\right],\ \ X=2\operatorname{Re}\left[s_1\Phi(z_1)z_1^{'}+s_2\Psi(z_2)z_2^{'}\right],$$
$$u'=2\operatorname{Re}\left[p_1\Phi(z_1)z_1^{'}+p_2\Psi(z_2)z_2^{'}\right],\ \ v'=2\operatorname{Re}\left[q_1\Phi(z_1)z_1^{'}+q_2\Psi(z_2)z_2^{'}\right], \tag{2}$$

where $u'=du/ds,\ v'=dv/ds$, $p_j=\alpha_{11}s_j^2-\alpha_{16}s_j+\alpha_{12},\ q_j=\alpha_{12}s_j-\alpha_{26}+\alpha_{22}/s_j$, $z_j^{'}=dx/ds+s_j dy/ds$, ds are a differential of arc at Γ .

We will use integral identities for complex potentials in the form [22, 23]

$$\Phi(z_1)=\int_L\left[P\Phi_3(z_1,t_1)+Q\Phi_4(z_1,t_1)\right]ds+\Phi_S(z_1),$$
$$\Psi(z_2)=\int_L\left[P\Psi_3(z_2,t_2)+Q\Psi_4(z_2,t_2)\right]ds+\Psi_S(z_2), \tag{3}$$

where $ds=\sqrt{(d\xi)^2+(d\eta)^2}$, $t_k=\xi+s_k\eta,\ k=1,2$, $(\xi,\eta)\in L$, $P=X, Q=Y$ are unknown projections of the stress vector on the boundary of inclusions at $(x,y)\in L$

$$\Phi_j=\frac{A_j}{t_1-z_1},\ \Psi_j=\frac{B_j}{t_2-z_2},\ j=3,4, \tag{4}$$

$$A_3=-\frac{ip_1}{2\pi\Delta_1},\ \ A_4=-\frac{iq_1}{2\pi\Delta_1},\ B_3=-\frac{ip_2}{2\pi\Delta_2},\ \ B_4=-\frac{iq_2}{2\pi\Delta_2},\ \Delta_j=\Delta'(s_j),\ j=1,2.$$

The sought solution must satisfy the conditions of balance of inclusions

$$\int_{L_j}Pds=X_j,\ \int_{L_j}Qds=Y_j,\ \int_{L_j}(Qy-Px)ds=M_j\ \ ,j=1,...,J. \tag{5}$$

We use the third condition (5) when considering the problem with unknown angles of rotation of inclusions ω_j (then the value of moments M_j are considered given).

After substituting the solution (3) into the boundary conditions (2), a system of integral equations for determining the unknown functions P, Q is obtained in the form [22]

$$2\operatorname{Re}\left[p_1\Phi(z_1)z_1^{'}+p_2\Psi(z_2)z_2^{'}\right]=g_1^{'}/2,\ \ 2\operatorname{Re}\left[q_1\Phi(z_1)z_1^{'}+q_2\Psi(z_2)z_2^{'}\right]=g_2^{'}/2, \tag{6}$$

where functions with waves at the top are determined by formulas (3), in which the Cauchy integrals are considered in the sense of the principal value.

We apply the method of mechanical quadrature to solve integral equations (6). Let us first consider the case of one inclusion. The parametric equation of its boundary can be written in the form $x=\alpha(\theta),\ y=\beta(\theta),\ 0<\theta\le 2\pi$. Using the quadrature formulas given in [22, 24], we obtain at the system of equations

$$H\sum_{k=1}^{N} s_k^{'}\left[P_k U^{(3)}\left(Z_\nu, T_k\right)+Q_k^{(j)}U^{(4)}\left(Z_\nu, T_k\right)\right]=-\omega y_\nu^{'}/2-U_{S\nu},\ \ \nu=1,\dots,N$$

$$H\sum_{k=1}^{N} s_k^{'}\left[P_k V^{(3)}\left(Z_\nu, T_k\right)+Q_k^{(j)}V^{(4)}\left(Z_\nu, T_k\right)\right]=\omega x_\nu^{'}/2-V_{S\nu}, \tag{7}$$

where $P_k=P(T_k),\ Q_k=Q(T_k);\ x_\nu^{'}=x'(Z_\nu),\ y_\nu^{'}=y'(Z_\nu);\ T_k$ and Z_ν are points with coordinates (ξ_k,η_k) and (x_ν, y_ν); $U^{(j)}(Z_\nu,T_k),\ V^{(j)}(Z_\nu,T_k)$ are derivatives of the displacement vector $\left(du/ds, dv/ds\right)$ on an arc L at $x=x_\nu, y=y_\nu$ and $\xi=\xi_k,\ \eta=\eta_k$, which are determined by formulas (2) by potentials (9) Φ_j,Ψ_j (j=3.4); coefficients $\left(U_{S\nu}, V_{S\nu}\right)$ are derivatives of the vector of displacements on the arc L at the point $\left(x_\nu,\ y_\nu\right)$, which correspond to the potentials Φ_S,Ψ_S. Here N is the selected number of nodal points, $\xi_k=\alpha(\theta_k),\ \eta_k=\beta(\theta_k),\ x_\nu=\alpha(\tau_\nu),$ $y_\nu=\beta(\tau_\nu),\ \theta_k=Hk,\ \tau_\nu=\theta_\nu-H/2,\ H=2\pi/N,\ s_k^{'}=s'(\theta_k),\ s'(\theta)=\sqrt{\alpha'(\theta)^2+\beta'(\theta)^2},$ $x'=\alpha'/s', y'=\beta'/s'$. It has been proved [22] that the systems of equations written in both lines (7) are linearly dependent. After removing one equation from them, let's supplement this system with equations that follow from the first two equilibrium conditions (5)

$$H\sum_{k=1}^{N} P_k s_k^{'}=X_1,\qquad H\sum_{k=1}^{N} Q_k s_k^{'}=Y_1. \tag{8}$$

As a result, we get a closed system of equations (7) and (8) for determining the unknowns $P_k, Q_k\ \ (k=1,\dots,N)$ at a given angle of rotation ω.

With an unknown angle of rotation of the inclusion, we additionally use the third equation (5), from which we obtain

$$H\sum_{k=1}^{N} s_k^{'}(P_k\eta_k-Q_k\xi_k)=M_1. \tag{9}$$

Consider a plate containing a system of inclusions, the boundaries of which are described by equations $\xi=\alpha_j\left(\theta\right),\ \eta=\beta_j\left(\theta\right),\ \ 0<\theta\le 2\pi,\ \ j=1,\dots,J$.

Then the system of equations (7) will be written

$$H\sum_{j=1}^{J}\sum_{k=1}^{N} s'\left(T_k^{(j)}\right)\left[P_k^{(j)}U^{(3)}\left(Z_\nu^{(i)},T_k^{(j)}\right)+Q_k^{(j)}U^{(4)}\left(Z_\nu^{(i)},T_k^{(j)}\right)\right]=-\omega_i y'\left(Z_\nu^{(i)}\right)/2-U_S\left(Z_\nu^{(i)}\right),$$

$$H\sum_{j=1}^{J}\sum_{k=1}^{N} s'\left(T_k^{(j)}\right)\left[P_k^{(j)}V^{(3)}\left(Z_\nu^{(i)},T_k^{(j)}\right)+Q_k^{(j)}V^{(4)}\left(Z_\nu^{(i)},T_k^{(j)}\right)\right]=\omega_i x'\left(Z_\nu^{(i)}\right)/2-V_S\left(Z_\nu^{(i)}\right), \tag{10}$$

where $i=1,...,J,\ \nu=1,...,N,$ $P_k^{(j)}=P(T_k^{(j)}),\ Q_k^{(j)}=Q(T_k^{(j)});\ T_k^{(j)}$ and $Z_\nu^{(i)}$ are points with coordinates $\left(\alpha_j(\theta_k),\beta_j(\theta_k)\right)$ and $\left(\alpha_i(\tau_\nu),\ \beta_i(\tau_\nu)\right)$.

In (10) with fixed values $i\,(i=1,...,J)$ and $1\le\nu\le N$ the equations are linearly dependent. We remove one of the equations from them and replace them with equilibrium equations

$$H\sum_{k=1}^{N}P_k^{(j)}s'\left(T_k^{(j)}\right)=X_j,\ \ H\sum_{k=1}^{N}Q_k^{(j)}s'\left(T_k^{(j)}\right)=Y_j,\ j=1,...,J\ . \tag{11}$$

For unknown angles of rotation of the inclusions, we add the equation

$$H\sum_{k=1}^{N}s'\left(T_k^{(j)}\right)\left[P_k^{(j)}y\left(T_k^{(j)}\right)-Q_k^{(j)}x\left(T_k^{(j)}\right)\right]=M_j,\quad j=1,...,J. \tag{12}$$

3. CALCULATION OF DISPLACEMENTS IN THE PLATE

When considering the problem with rivets, it is necessary to determine the displacement at arbitrary points of the plate. The displacements in the plate are determined by formulas [22]

$$u=2\,\mathrm{Re}\left[p_1\varphi\left(z_1\right)+p_2\psi\left(z_2\right)\right],\ \ v=2\,\mathrm{Re}\left[q_1\varphi\left(z_1\right)+q_2\psi\left(z_2\right)\right], \tag{13}$$

where

$$\begin{aligned}\varphi(z_1)&=-\int_L\left(A_3P+A_4Q\right)\ln\left(z_1-t_1\right)ds+\int\Phi_S(z_1)dz_1,\\ \psi(z_2)&=-\int_L\left(B_3P+B_4Q\right)\ln\left(z_2-t_2\right)ds+\int\Psi_S(z_2)dz_2.\end{aligned} \tag{14}$$

On the basis of these formulas, we obtain a relationship for determining the displacements at an arbitrary point of the plate Z with inclusions of an elliptical shape with semi-axes a_j,b_j and centers in C_j

$$u(Z)=\sum_{j=1}^{J}\sum_{k=1}^{N}\left(P_k^{(j)}u_{Pk}^{(j)}(Z)+Q_k^{(j)}u_{Qk}^{(j)}(Z)\right)+u_S(Z),\ \ v(Z)=\sum_{j=1}^{J}\sum_{k=1}^{N}\left(P_k^{(j)}v_{Pk}^{(j)}+Q_k^{(j)}v_{Qk}^{(j)}\right)+v_S(Z),$$

where $u_S(Z),v_S(Z)$ are displacements in a continuous plate under the applied external load. Functions $u_{Pk}^{(j)}(Z),u_{Qk}^{(j)}(Z),v_{Pk}^{(j)}(Z),v_{Qk}^{(j)}(Z)$ are determined based on quadrature formulas for integrals (14).

4. CALCULATION OF DISPLACEMENTS IN THE RODS

Assuming that the plate is reinforced K rods that are parallel to the axis *Ox*. Let there be rivets with numbers $I_k < j \le J_k$ on the *k*-th rod. There are points $\left(c_j, d_j\right)$ at it, to which the centers of inclusions correspond, forces (X_{Bj}, Y_{Bj}) and moments M_{Bj} act, moreover $X_{Bj} = -\delta X_j, Y_{Bj} = -\delta Y_j, M_{Bj} = -\delta M_j$, where δ is plate thickness. Displacements in the *k*-th rod due to this load are determined by the formulas [25]

$$E_B S_B U_k\left(x,y\right) = E_B S_B (D_{Bk} - \omega_{Bk} y) - \sum_{j=I_k}^{J_k} X_{Bj}\left(x - c_j\right)_+ ,$$

$$E_B I_B V_k\left(x,y\right) = E_B I_B (A_{Bk} + \omega_{Bk} x) - \sum_{j=I_k}^{J_k} \frac{M_{Bj}\left(x - c_j\right)_+^2}{2!} + \sum_{j=I_k}^{J_k} \frac{Y_{Bj}}{3!}\left(x - c_j\right)_+^3 , \qquad (15)$$

where $A_{Bk}, D_{Bk}, \omega_{Bk}$ are unknown constants, E_B, I_B and S_B are modulus of elasticity, moment of inertia and cross-sectional domain of rods, $\left(t\right)_+ = t, t > 0,\ \left(t\right)_+ = 0, t \le 0.$

The equilibrium conditions of the forces applied to each of the rods have the form

$$\sum_{j=I_k}^{J_k} X_j = 0, \sum_{j=I_k}^{J_k} Y_j = 0, \sum_{j=I_k}^{J_k}\left(c_j Y_j + M_j\right) = 0, k = 1,...,K . \qquad (16)$$

Let us further assume that the rivets have a symmetrical shape relative to their centers with the half-lengths of the axes of symmetry *a*,*b*. The center of the k-th inclusion is moved along the coordinate axes, and the inclusion is turned by an angle ω_k.

Let us write down the compatibility conditions of deformations of the rod and the plate in the form

$$u(C_{Rj}) = U_B(C_j), v(C_{Vj}) = V_B(C_j), \omega_j = V_B^{'}(C_j), j = 1,...,J, \qquad (17)$$

where C_{Rj} is points with coordinates $(c_j + a, d_j)$, C_{Vj} is points with coordinates $\left(c_j, d_j + b\right)$.

Substituting formulas (15) into conditions (17), we obtain the equation

$$D_{Bk} - \omega_{Bk} d_m + \frac{\delta}{E_B S_B}\sum_{j=I_k}^{J_k} X_j\left(c_m - c_j\right)_+ = \sum_{j=1}^{J}\sum_{k=1}^{N}\left[P_k^{(j)} u_{Pk}^{(j)}\left(C_{Rm}\right) + Q_k^{(j)} u_{Qk}^{(j)}\left(C_{Rm}\right)\right] + u_S\left(C_{Rm}\right),$$

$$A_{Bk} + \omega_{Bk} c_m + \frac{\delta}{E_B I_B}\sum_{j=I_k}^{J_k}\frac{M_j\left(c_m - c_j\right)_+^2}{2!} - \frac{\delta}{E_B I_B}\sum_{j=I_k}^{J_k}\frac{Y_j}{3!}\left(c_m - c_j\right)_+^3 = \sum_{j=1}^{J}\sum_{k=1}^{N}\left[P_k^{(j)} v_{Pk}^{(j)}\left(C_{Vm}\right) + Q_k^{(j)} v_{Qk}^{(j)}\left(C_{Vm}\right)\right] + v_S\left(C_{Vm}\right), \quad (18)$$

$$\omega_{Bk} + \frac{\delta}{E_B I_B}\sum_{j=I_k}^{J_k} M_j\left(c_m - c_j\right)_+ - \frac{\delta}{E_B I_B}\sum_{j=I_k}^{J_k}\frac{Y_j}{2}\left(c_m - c_j\right)_+^2 = \omega_m, \quad I_k < m \le J_k, 1 \le k \le K$$

In this way, a complete system of equations (7,8,16,18) is obtained for finding unknown $P_n^{(j)}, Q_n^{(j)}\ (n=1,...,N);\ X_j, Y_j, M_j, \omega_j\ (j=1,...,J);\ A_{Bk}, D_{Bk}, \omega_{Bk}\ (k=1,...,K)$.

5. CALCULATION RESULTS

Calculations are made for isotropic and boron-epoxy plates. For an isotropic plate the Young's modulus was taken $E_S = 200GPa$; Poisson's ratio $\nu_S = 0.3$. For boron-epoxide: modulus of elasticity $E_x = 21GPa, E_y = 207GPa$; shear modulus $G = 7\ GPa$; Poisson's ratios $\nu_{yx} = 0.0304$, $\nu_{xy} = 0.3$ [26].

5.1. One row of rivets

Considered a rod connected by eight circular rivets of radius *R*, which are placed on the *Oy* axis. The distances between the centers of neighboring inclusions were taken as *5R*; the plate is stretched in the vertical direction by forces *p* (Fig. 1.a), the material of the plate is boron-epoxy.

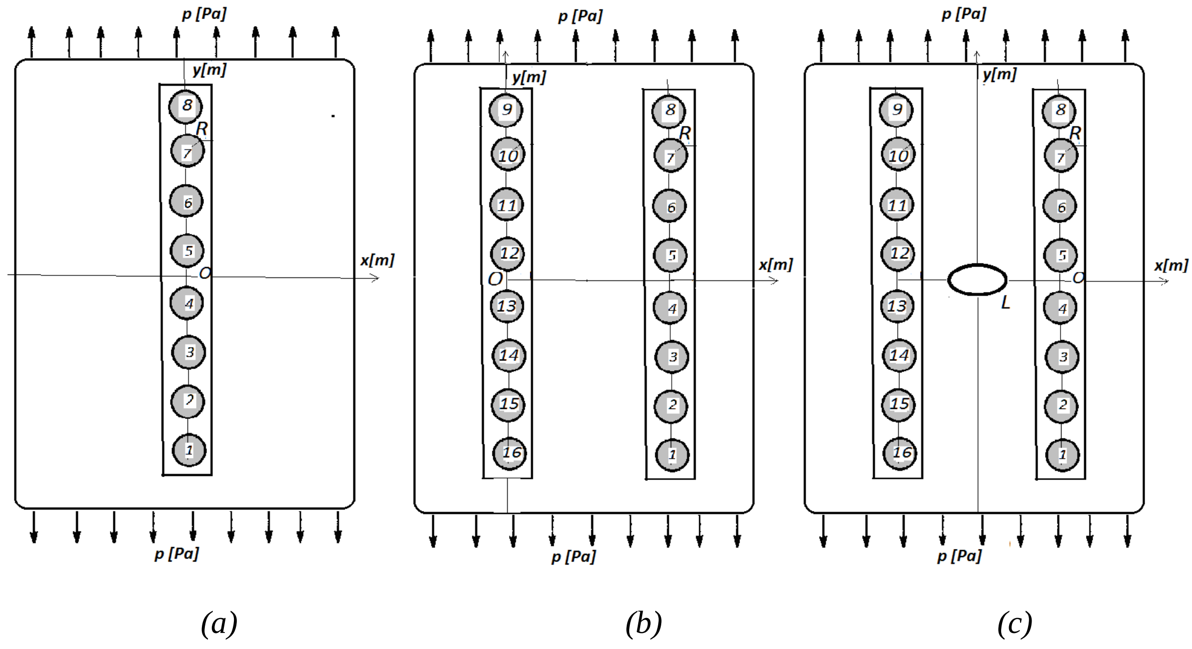


Fig. 1. The plate is attached to the rod with rivets

Figure 2 shows the relative normal stresses $\sigma = \sigma_n / p$ on the boundaries of the 1st inclusion (shown by solid lines) and the second inclusion (dashed lines) in the boron-epoxy plate. The value of the parameter is indicated near the curves, $\lambda = {E_s \delta R}/{E_B S_B}$ which characterizes the rigidity of the rod.

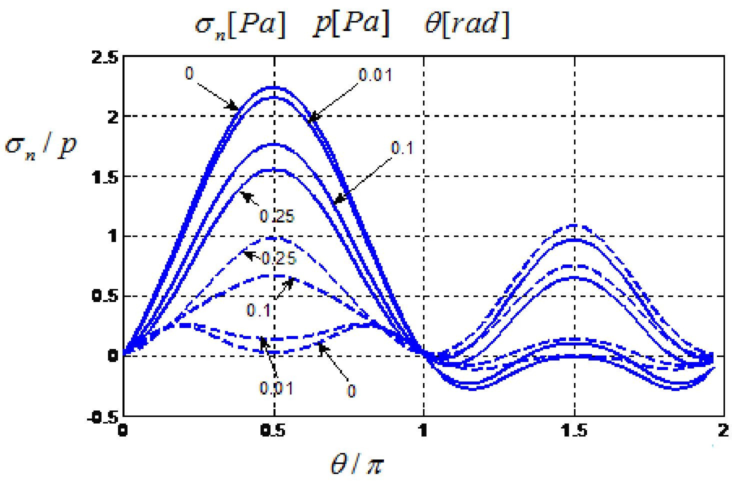


Fig. 2. Hoop stresses at the edge of the rivets

Stresses on horizontal middle lines between inclusions (σ_y / p) depending on the distance from the straight line on which the centers of inclusions are located are shown in Fig. 3 at $\lambda = 0.01$. Curves j=1,2,3,4 show stresses on lines placing between j and j+1 inclusions. The value j=-1 corresponds to the line placed below the first inclusion.

Stresses on horizontal middle lines between inclusions (σ_y / p) depending on the distance from the straight line on which the centers of inclusions are located are shown in Fig. 3 at $\lambda = 0.01$. Curves j=1,2,3,4 show stresses on lines placing between j and j+1 inclusions. The value j=-1 corresponds to the line placed below the first inclusion.

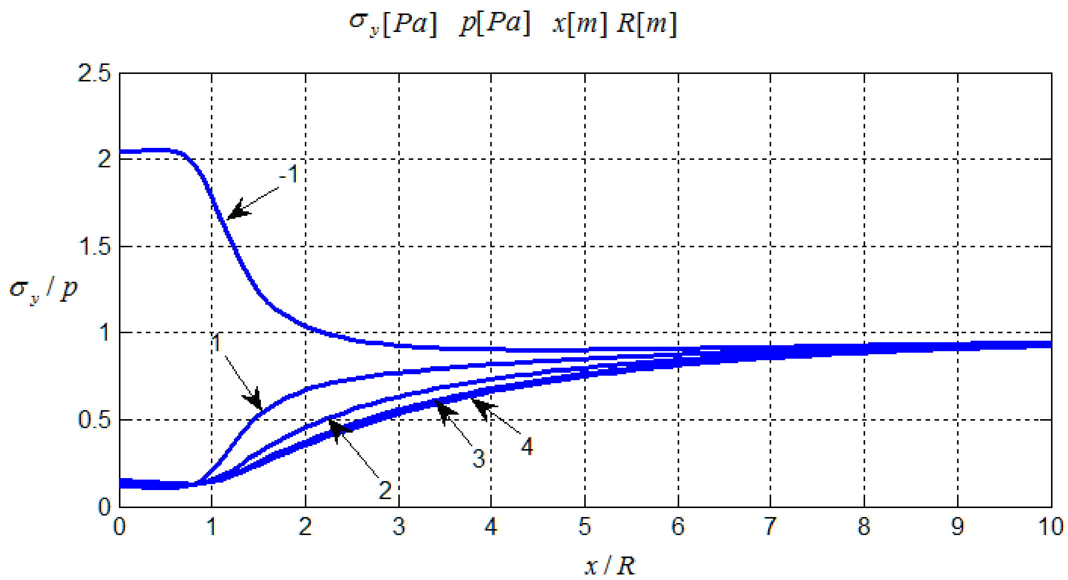


Fig. 3. Relative stresses between inclusions
(1 is between the first and second, 2 is between the second and third, 3 is between the third and fourth, 4 is between the fourth and fifth)

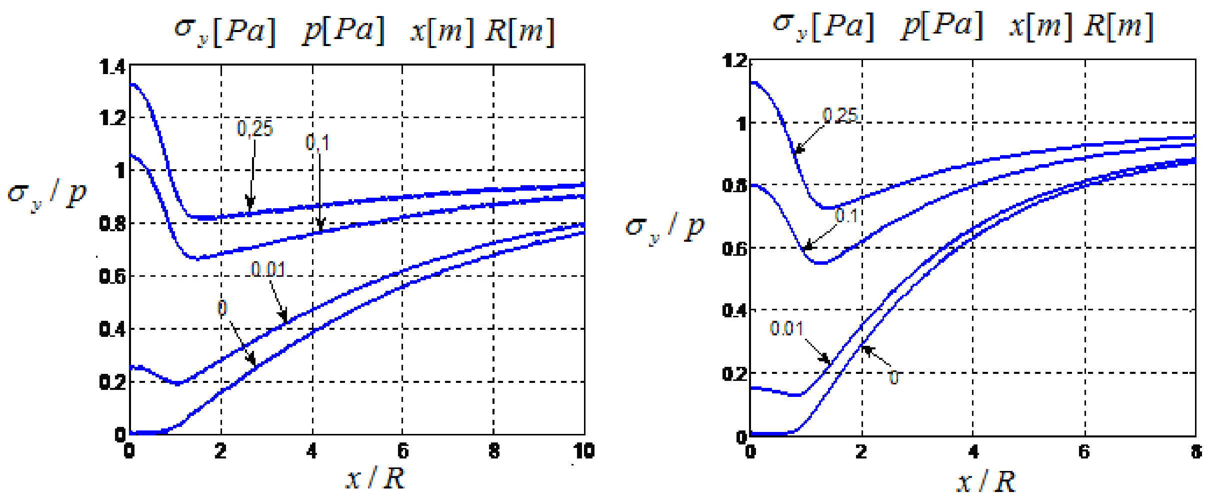
It can be seen that low stresses occur between the inclusions, which are smaller than the applied forces p. The smallest stresses occur directly between the inclusions. Stresses decrease in magnitude when approaching the center of the rod. The stresses below the first rivet at a distance of *2.5R* are significantly larger than those between the rivets.

Reinforcement, as a rule, is carried out in order to reduce stresses in the domain near the center of the rod. Therefore, the relative stresses are in the middle between the inclusions (on the *Ox* axis) at different values of the parameter λ shown in Fig. 4.a.

The smallest stresses occur in the middle between inclusions with an absolutely rigid rod (at $\lambda = 0$). When the rigidity of the rod decreases, the stresses increase, remaining smaller in magnitude than the applied load. At the values of the rigidity parameter $0.1 < \lambda < 0.25$ there is a domain centered at $x \approx 1.5R$, in which the stresses are minimal.

The value of relative forces $P = Y/(pR)$, which are attached to centers 1-4 of inclusions are given in Table 1. On the remaining inclusions, the forces act symmetrically (they have opposite signs).

It can be seen that the maximum values of the forces occur on the two extreme inclusions (rivets), while they significantly exceed the forces in the neighboring inclusions, and the forces in the central inclusions are close to zero.



(a) *(b)*

Fig. 4. Relative stresses in the plate at the center line between the rivets

Tab. 1

Values of relative forces *P* applied to inclusions as a result of interaction with the rod

$N \backslash \lambda$	0	0.01	0.1	0.25
1	7.1012	6.5114	3.7678	2.2393
2	1.4586	1.2852	0.5609	0.2477
3	0.7242	0.6322	0.2598	0.1090
4	0.2248	0.1954	0.0782	0.0321

Considered a rod riveted with 16 rivets to a boron-epoxy plate. The calculated relative stresses between the central inclusions are shown in Fig. 1.b. Values of relative forces and maximum stresses $\sigma = \sigma_n / p$ at the boundary of inclusions are given in Table 2.

Tab. 2

Values of relative forces and maximum stresses on the first inclusions, boron-epoxy plate

λ	0		0.01		0.1		0.25	
N	P	max σ	P	max σ	P	max σ	P	max σ
1	9.8690	2,979	8.6461	2.75	4.2659	1,958	2.3927	1.643
2	2.4832	0.389	2.0704	0.382	0.7348	0.868	0.2980	1.169
4	1,1964	0.197	0.9697	0.235	0.2905	0.998	0.1034	1.287
6	0.5793	0.116	0.4627	0.254	0.1270	1.055	0.0423	1.335
8	0.1104	0.056	0.0876	0.263	0.0232	1,073	0.0076	1.347

It can be seen that an increase in the length of the rod and the number of rivets causes to an increase in: the value of forces on the extreme rivets; normal stresses on these rivets; stresses between the central rivets.

Let's consider 2 ways to reduce these stresses. In the first, we will reduce the distance between the rivets, taking it as *3R*. It was established that reducing the distance between the rivets allows reducing the forces that occur on the extreme rivets and the maximum stresses on their borders. At the same time, the stresses between the central rivets increased near them and decreased at long distances from the rod.

In the second method, the influence of the shape of the rivets on the stress distribution is considered. For this purpose, elliptical rivets with different semi-axes (*nR*,*mR*) for boron-epoxy plate were considered. The calculated stresses at the boundary of the first rivet for a rigid rod at different values of parameters *n*, *m* (which are indicated near the curves) are shown in Fig. 5.

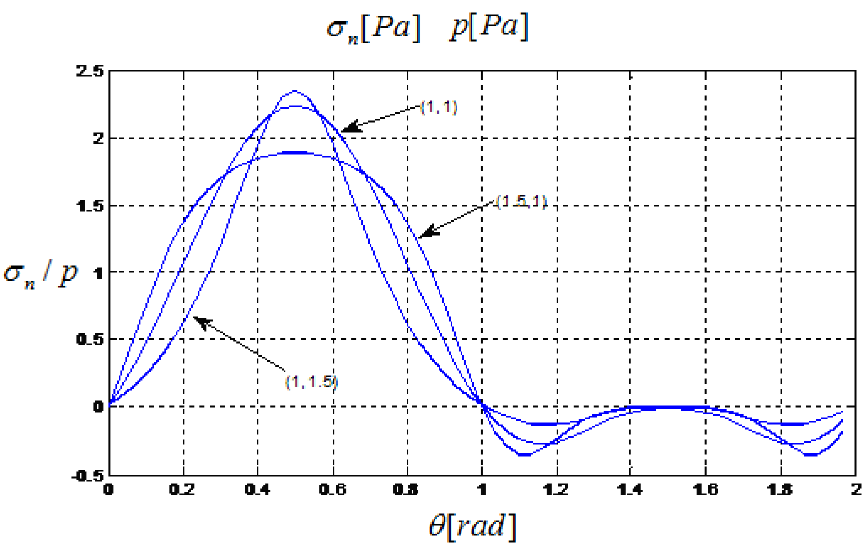


Fig. 5. Relative normal stresses at the boundary of the first elliptical rivet

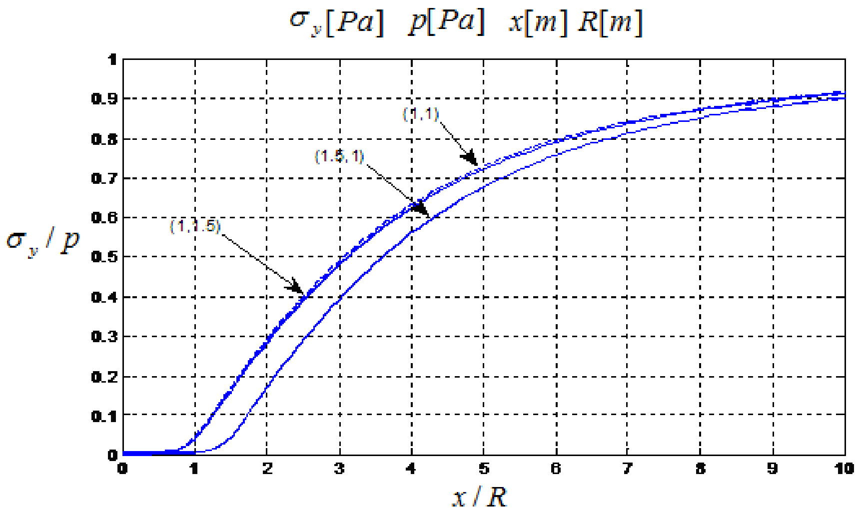


Fig. 6. The stresses between the central elliptical rivets

It can be seen from Fig. 5 that on vertically flattened rivets of an elliptical shape with semi-axes (*1.5R,R*), the stresses have significantly decreased. At the same time, near the elongated rivets with semi-axes (*R, 1.5R*), the stresses were the largest.

The stresses between the two central rivets are shown in Fig.6From Fig. 5, 6 it can be seen that the use of rivets of an elliptical shape with semi-axes (*1.5R,R*) allows reducing the stress in the plate.

5.2. Two rows of rivets

The case where the plate is reinforced by two rods is considered. In the first rod, rivets are placed on the line $x=0$, the distance between adjacent rivets is equal to $5R$ (Fig. 1.b). The second row of rivets is placed symmetrically on the line $x=10R$.

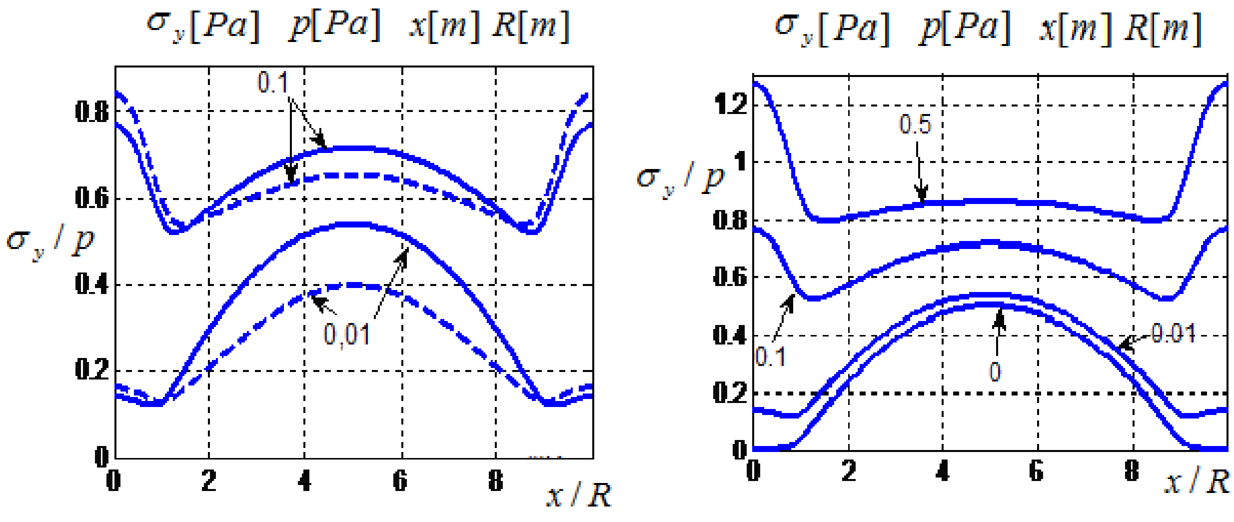
An isotropic plate is considered. The calculated values of the relative forces applied to the first four inclusions and the maximum normal forces on the boundaries of these inclusions are given in Table 3. In other inclusions, forces and stresses are symmetric or antisymmetric with respect to the given ones.

The relative stresses between the central rivets are shown in Fig. 7a.

Tab. 3

Maximum stresses on the boundaries of inclusions and values of forces applied to them

λ	0		0.01		0.1		0.5	
N	P	max σ	P	max σ	P	max σ	P	max σ
1	6,6191	2,101	6,1032	2,032	3,6224	1,7049	1,3194	1,4149
2	1,3335	0,239	1,1816	0,252	0,5291	0,7226	0,1072	1,2408
3	0,6544	0,134	0,5743	0,163	0,2419	0,7669	0,0448	1,2805
4	0,2018	0,071	0,1764	0,148	0,0723	0,7813	0,0128	1,2923



(a) *(b)*

Fig. 7. Stresses in an isotropic plate with two rods between central rivets

Based on the calculations, it follows that the greatest stresses occur on the extreme rivets, which can negatively affect the strength at high operating loads. Let us consider one of the ways in which these tensions can be reduced. For this purpose, let's increase the size of the extreme rivets twice. Figure 8 shows the normal stresses on the first rivet. Calculations were performed for two values of the rod rigidity parameter λ : 0.01 and 0.1. Solid lines show data for rivets of the same size, and dashed lines show data for increased sizes of extreme rivets

The results of stress calculations between the central rivets are shown in Fig. 7b. From the results of the calculations, it follows that increasing the size of the extreme rivets allows you to significantly reduce the stress near them and at the same time reduce the stress in the central domain between the rods, in which the stress concentrators can be placed.

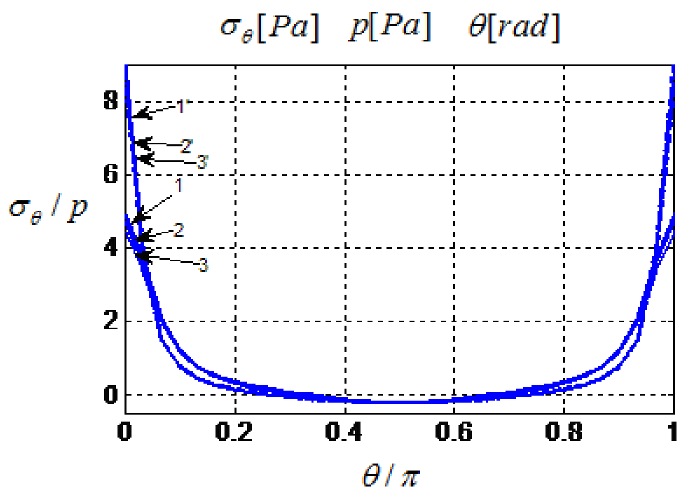


Fig. 8. Relative normal stresses at the boundary of
the first rivet of enlarged dimensions, boron-epoxy plate

Considered a plate with two rods, each of which contains 8 rivets. In the central domain (Fig. 1.c) between the rods, the plate is weakened by an elliptical hole with semi-axes *(a,b).* Solving this problem is performed out on the basis of integral equations, which are built on the basis of boundaries for potentials of the form (3) and Green's solutions $\Phi_j, \Psi_j, j=3,4$. Functions $\Phi_j, \Psi_j, j=3,4$ are replaced by Green's solutions, which for the case of an elliptical hole are given in [27]. Holes of different shapes are considered. The calculated relative hoop stresses on circular holes with radii *kR* at *k=1,2,3* are shown in fig. 9 with solid lines (the value of k is indicated near the curves).

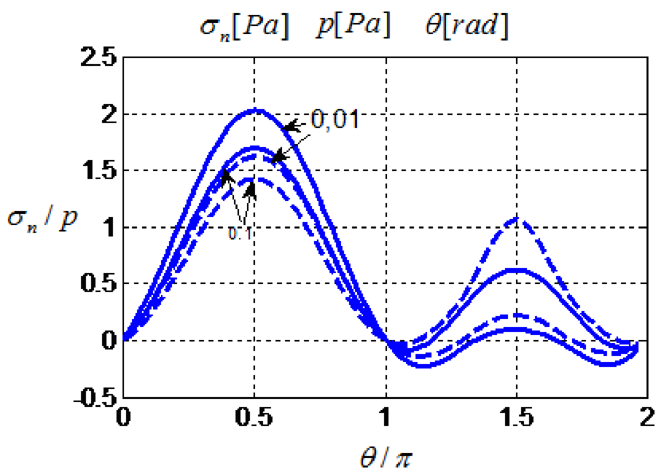


Fig. 9. Relative hoop stresses on the boundaries of circular (curves 1-3) and elliptical holes (curves 1'-3')

Here, the stresses are given depending on the angular coordinate on the upper semicircle. Dashed lines show the stresses for an elliptical hole with semi-axes *(R,0.5R),(2R,R),(3R,1.5R)* (curves 1',2',3'). Accepted: the distance between the rivets is *5R*, the distance between the rods is *10R*.

It can be seen that the presence of reinforced rods made it possible to reduce the stress concentration (in homogeneous boron-epoxy plates, the stress concentration coefficient is equal to 6.937 for a circular hole and 12.874 for an elliptical hole with a half-axis ratio of ½ [28]). At the same time, the hoop stresses decrease somewhat when the sizes of the circular holes increase. With different sizes of elliptical holes, the stresses on their boundaries are practically the same, while when their sizes increase, the stresses also decrease slightly.

6. CONCLUSIONS

An approach to the study of stresses and deformations in composite plates reinforced by rods, which are attached with rivets, is proposed. Plates are considered both solid and weakened by a hole, and rivets - as rigid inclusions. Determination of stresses and deformations in the plate is carried out by the method of singular integral equations. To determine the unknown forces that act on the inclusion (rivets), additional equations are written, which are obtained from the compatibility conditions of deformations of the plate and the rod. At the same time, obtained relations were used to determine the displacements of inclusions. Examples of stress calculation near rivets of circular and elliptical shape are given.

Stresses in a solid plate reinforced by an elastic rod with and 16 rivets were investigated. The values of the forces acting on the rivets, the stress on the boundaries of the inclusions (rivets) and in the plate, depending on the rigidity of the rods, were determined.

Similar studies were performed for plates reinforced by two rods, each of which is attached with 8 rivets. It was established that in all the considered cases the greatest forces occur on the extreme rivets. Methods that allow reducing these forces are considered: reducing the distance between rivets; use of elliptical rivets; increase in the size of extreme rivets. The influence of two riveted rods on the reduction of stress concentration near the elliptical hole placed between them was studied.

Notations parameters, variables and functions

J is number of rivets (inclusions).
δ_j [m] and ω_j [rad] are displacement and rotation of inclusion with number *j*.
α_{ij} [Pa^{-1}] are elastic compliances which are included in Hooke's law.
(X_j, Y_j) [N] and M_j [Nm] are the principal vector and moment applied to the inclusion with the number *j*.
$s_j, j = 1, 2$ are roots of characteristic equation.
u, *v* [m] are displacement of plate points.
σ_x, σ_y, τ_{xy} [Pa] are stresses in the plate.
ε_x, ε_y, γ_{xy} are plate deformations.
$\varphi, \psi, \Phi, \Psi -$ are complex potentials of Lekhnitskyi.
s [m] is arc coordinate.
θ [rad] is angular coordinate on the edge of the rivet
$E_B[Pa], I_B[m^4]$ and $S_B[m^2]$ and is modulus of elasticity, moment of inertia and cross-sectional area of the rod.
E_x, E_y and G [Pa] are modulus of elasticity and shear of the composite material.
ν_{xy}, ν_{yx} are Poisson's ratios.
R [m] is rivet radius.
p [Pa] is the force used to stretch the plate.
$\lambda -$ is a dimensionless parameter that characterizes the stiffness of the rod.
P [N] is value of forces applied to rivets.
σ are relative stress values in the plate at the inclusion boundary.

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